1 Additional Exercises: Cosets

- Determine the index \([Z : nZ]\).
- Prove directly that distinct cosets do not overlap.
- Prove every group whose order is a power of a prime \(p\) contains an element of order \(p\).
- Give an example showing that left cosets and right cosets of \(GL(2, R)\) in \(GL(2, C)\) are not always equal.
- Let \(H, K\) be subgroups of a group \(G\) of orders 3, 5 respectively. Prove \(H \cap K = \{e\}\).
- Do
  1. Let \(G\) be an abelian group of odd order. Prove the map \(\phi: G \rightarrow G\) defined by \(\phi(x) = x^2\) is an automorphism.
  2. Generalize the above result.
- Let \(W\) be additive subgroup of \(R^m\) of solutions of a system of homogeneous linear equations \(AX = 0\). Show the solutions of a non-homogeneous system \(AX = B\) form a coset of \(W\).
- Do:
  1. Prove that every subgroup of index 2 is normal.
  2. Give an example of a subgroup of index 3 that is not normal.
- Let \(G, H\) be the following subgroups of \(GL(2, R)\):

\[
G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \right\}, \quad H = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \right\}, \ x > 0.
\]

An element of \(G\) can be represented by a point in the \((x, y)\) plane. Draw the partitions of the plane into left and into right cosets of \(H\).