Double Integral Example Worksheet

Double Integrals over general regions in $x, y$ coordinates

Sketch regions too

1. $\int_0^4 \int_0^{4-x} xy \, dy \, dx$
   - Inner: $\int_0^{4-x} xy \, dy = \frac{1}{2} x y^2 \bigg|_{y=0}^{y=4-x} = \frac{1}{2} x (4-x)^2$
   - Completion: $\int_0^4 \frac{1}{2} x (4-x)^2 \, dx = \frac{1}{2} \int_0^4 (x^3 - 8x^2 + 16x) \, dx = \frac{32}{3}$

2. $\iint_D (x + y) \, dA$ where $D$ is the triangle with vertices $(0,0), (0,2), (1,2)$
   - $\int_0^1 \int_{2-2x}^{2x} (x + y) \, dy \, dx = \int_0^1 \int_{2-2x}^{2x} (x + y) \, dy \, dx = \frac{5}{2}$.

3. $\iint_D 48xy \, dA$ where $D$ is the region bounded by $y = x^3$ and $y = \sqrt{x}$
   - $\int_1^2 \int_{x^3}^{\sqrt{x}} 48xy \, dy \, dx = \int_1^2 \int_{x^3}^{\sqrt{x}} 48xy \, dy \, dx = 5$

Reverse order of integration.

1. $\int_0^{2x} \int_y^x e^{x-y} \, dy \, dx = \int_{x=0}^{2x} \int_{y=x}^{y=\frac{1}{2}x} e^{x-y} \, dy \, dx + \int_0^{2x} \int_{y=\frac{1}{2}x}^y e^{y-x} \, dy \, dx$

2. $\int_0^{2\sqrt{3}} \int_{\sqrt{16-y^2}}^1 1 \, dy \, dx = \int_0^{2\sqrt{3}} \int_{\sqrt{16-y^2}}^1 1 \, dy \, dx + \int_0^{2\sqrt{3}} \int_0^{\sqrt{16-x^2}} 1 \, dy \, dx = \frac{2}{3} \sqrt{3} + \frac{8}{3} \pi$

3. $\int_0^2 \int_{-6x}^2 f(x,y) \, dy \, dx = \int_0^2 \int_{-6x}^2 f(x,y) \, dy \, dx + \int_0^2 \int_{-6x}^{f(x,y)} f(x,y) \, dy \, dx$

4. $\int_0^2 \int_x^3 f(x,y) \, dy \, dx + \int_2^8 \int_x^8 f(x,y) \, dy \, dx = \int_0^2 \int_{x=\frac{1}{2}}^y f(x,y) \, dy \, dx$

Find Volume of solid

1. Tetrahedron in first octant bounded by coordinate planes and $z = 7 - 3x - 2y$.
   - $\int_0^7 \int_0^{\frac{7-3y}{2}} (7 - 3x - 2y) \, dy \, dx = \int_0^7 \int_0^{\frac{7-3x}{2}} (7-3x-z) \, dz \, dx = \int_0^7 \int_0^{\frac{7-2y}{3}} (7-2y-z) \, dz \, dy = \frac{343}{36}$

2. Solid inside both the sphere $x^2 + y^2 + z^2 = 3$ and paraboloid $2z = x^2 + y^2$.
   - $\int_0^3 \int_{\sqrt{3-x^2-y^2}}^{\sqrt{x^2+y^2}} \left( \sqrt{3-x^2-y^2} - \frac{x^2+y^2}{2} \right) \, dy \, dx = 2\sqrt{3}\pi - \frac{1}{4}\pi - \frac{4}{3}\sqrt{2}\pi$

Double Integrals using polar coordinates

Direct Computations in polar coordinates

1. Compute $\int_0^{\pi/2} \int_0^1 r e^{-r^2} \, dr \, d\theta$
   - Inner: $\int_0^1 r e^{-r^2} \, dr = -\frac{1}{2} e^{-9} + \frac{1}{2} e^{-1}$ Using $u = -r^2$ and $du = -2r \, dr$
   - Completion: $\int_0^{\pi/2} \int_0^1 r e^{-r^2} \, dr \, d\theta = -\frac{1}{4} e^{-9}\pi + \frac{1}{4} e^{-1}\pi$

2. Find the area bounded by the cardioid $r = 1 + \sin \theta$.  

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\[ \iint_{D} 1 \, dA = \int_{0}^{2\pi} \int_{0}^{1+\sin(\theta)} r \, dr \, d\theta = \frac{3}{2} \pi \]

3. Find the area bounded by one leaf of the rose \( r = 4 \cos \theta \)

\[ \iint_{D} 1 \, dA = \int_{0}^{\pi} \int_{0}^{4 \cos(\theta)} r \, dr \, d\theta = 4\pi \]

4. Find area inside both \( r = 1 \) and \( r = 2 \sin \theta \).

\[ \iint_{D} 1 \, dA = 2 \left( \int_{0}^{\pi} \int_{0}^{1} r \, dr \, d\theta + \int_{\pi}^{2\pi} \int_{0}^{2 \sin(\theta)} r \, dr \, d\theta \right) = \frac{1}{2} \pi + \sqrt{3} \]
Convert from Cartesian \((x, y)\) to polar coordinates before integrating

1. Find \(\iint_D f(x, y) \, dA\) where \(D\) is the region bounded by the \(x\)-axis, the line \(y = x\) and the circle \(x^2 + y^2 = 1\).
\[
\iint_D f(x, y) \, dA = \int_0^\pi \int_0^1 f(r \cos(\theta), r \sin(\theta)) \, r \, dr \, d\theta
\]

2. Find the volume of the solid bounded by the paraboloid \(z = 4 - x^2 - y^2\) and the \(xy\)-plane.
\[
V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2) \, dy \, dx = \int_0^{2\pi} \int_0^2 (4 - r^2) \, r \, dr \, d\theta = 8\pi
\]

3. Find the volume inside the sphere \(x^2 + y^2 + z^2 = 25\) and outside the cylinder \(x^2 + y^2 = 9\).
\[
V = \iiint_D \left(\sqrt{25 - x^2 - y^2} - 0\right) \, dA = 2 \int_0^{2\pi} \int_0^3 \sqrt{25 - r^2} \, r \, dr \, d\theta = \frac{256}{3}\pi
\]

4. Find the volume inside the sphere \(x^2 + y^2 + z^2 = 25\) and outside the cylinder \((x - 1)^2 + y^2 = 1\).
   [This is a project problem but a hint is to write the equation of the cylinder in polar coordinates.]