Problems

1. In the following you will explore some of the properties of the Gamma Function $\Gamma(x)$. One of the tools you will need is the fact (proven later in the semester) that if $a$ is a positive constant then

$$\lim_{x \to \infty} \frac{x^a}{e^x} = 0.$$ 

Another tool you will need is the fact that the improper integral

$$\int_0^\infty e^{-x^2} \, dx$$ 

converges to the value $\sqrt{\pi}/2$. (This is proven in multivariable calculus.)

The Gamma function is defined as follows:

$$\Gamma : (0, \infty) \to R$$

$$\Gamma : x \mapsto \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt.$$ 

For example, note that

$$\Gamma(2) = \int_0^\infty t^{2-1}e^{-t} \, dt$$

$$= \int_0^\infty te^{-t} \, dt.$$ 

This is an improper integral that can be evaluated by using integration by parts.

(a) Directly evaluate the improper integral for $\Gamma(1)$.

(b) Use the substitution $t = u^2$ to evaluate the improper integral for $\Gamma(1/2)$.

(c) Use integration by parts to show that for any $x > 0$,

$$\Gamma(x + 1) = x\Gamma(x)$$
(d) Without directly evaluating any improper integrals, compute \( \Gamma(2) \), \( \Gamma(3) \), \( \Gamma(3/2) \), \( \Gamma(5/2) \), and \( \Gamma(-1/2) \).

(e) Explain why some people use "\( \frac{1}{2}! \)" when they refer to \( \Gamma(3/2) \). What would these people say when they refer to \( \Gamma\left(\frac{11}{6}\right) \)?