Directions: Whenever appropriate, use in-line citations, including page numbers and people consulted when you present information obtained from discussion, a text, notes, or technology. Only write on one side of each page.

“No, no, you’re not thinking, you’re just being logical.” -Niels Bohr, physicist (1885-1962)

Project Description

Discrete domain functions (sequences) have derivative formulas and rules that are analogous to the formulas and rules of interval domain functions. We will explore a few of them in this handout.

For example, consider the sequence given by the function \( a(n) = n^2 \) which is more precisely defined by

\[
\begin{align*}
a : &\quad \mathbb{N} \cup \{0\} \to \mathbb{R} \\
a : &\quad n \to a(n) = n^2
\end{align*}
\]

Then, \( D_n[a(n)] = \frac{a(n+1) - a(n)}{1} \) becomes

\[
\begin{align*}
D_n[a(n)] &= \frac{a(n+1) - a(n)}{1} \\
&= a(n+1) - a(n) \\
&= (n+1)^2 - n^2 \\
&= [(n+1) + n][(n+1) - n] \\
&= (2n+1)(1) \\
&= 2n + 1
\end{align*}
\]

Thus we can say that the line segment joining terms \( n \) and \( n+1 \) of this sequence has slope \( 2n + 1 \). As a specific example, the line segment joining terms 3 and 4, \( a(3) = 9, a(4) = 16 \) has slope \( m = 2(3) + 1 \) because \( \frac{16-9}{4-3} = 2(3) + 1 \).

So we now have a formula for discrete derivatives: \( D_n[n^2] = 2n + 1 \) and in a similar fashion we can compute \( D_n[n^3] = 3n^2 + 3n + 1 \) and \( D_n[n^4] = 4n^3 + 6n^2 + 4n + 1 \).

Here are a couple of ‘nicer’ examples:

1. First, let \( p \) be a positive integer and define \( n^p = n (n-1) (n-2) \cdots (n-p+1) \). As an example, \( n^3 = n (n-1) (n-2) \) and \( (n+1)^3 = (n+1) (n) (n-1) \). If we now compute the discrete derivative of the sequence \( a(n) = n^2 \) we get the “familiar” formula \( D_n[n^2] = 3n^2 \).

\[
\begin{align*}
D_n[a(n)] &= \frac{a(n+1) - a(n)}{1} \\
&= a(n+1) - a(n) \\
&= (n+1)^2 - n^2 \\
&= (n+1) (n) (n-1) - n (n-1) (n-2) \\
&= n (n-1) ((n+1) - (n-2)) \\
&= 3n (n-1) \\
&= 3n^2
\end{align*}
\]
2. Let \( a(n) = 3^n \). Then, \( D_n[3^n] = 2 \cdot 3^n \)

\[
D_n[a(n)] = \frac{a(n+1) - a(n)}{1}
\]

\[
= a(n+1) - a(n)
\]

\[
= 3^{n+1} - 3^n
\]

\[
= 3^n(3 - 1)
\]

\[
= 2 \cdot 3^n
\]

**Homework Problems**

1. Show

\[
D_n[n^2] = 2n^{1.5}
\]

\[
= 2n
\]

2. Show

\[
D_n[n^3] = 4n^3
\]

3. Show

\[
D_n[n^5] = 5n^4
\]

4. Show

\[
D_n[n^p] = pn^{p-1}
\]

5. Show

\[
D_n[2^n] = 2^n
\]

6. Show

\[
D_n[4^n] = 3 \cdot 4^n
\]

7. Show

\[
D_n[5^n] = 4 \cdot 5^n
\]

8. Show

\[
D_n[r^n] = (r - 1) r^n
\]