The Problems

**You may use technology for any problem other than the first.**

1. Solve the following system of equations by hand.

\[
\begin{align*}
  x_1 - x_2 - 2x_3 - x_4 &= -3 \\
  3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\
  2x_1 - 2x_2 - 3x_3 &= -2 
\end{align*}
\]

2. Do one of the following

   (a) Find the inverse of the matrix below or show that the inverse does not exist.

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 8 \\ 1 & 2 & 2 \end{bmatrix}
\]

   (b) Determine if the following collection of vectors in \( \mathbb{R}^4 \) are linearly independent or dependent.

\[
\begin{aligned}
  \vec{v}_1 &= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, & \quad \vec{v}_2 &= \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, & \quad \vec{v}_3 &= \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}
\end{aligned}
\]
3. Do one of the following

(a) Find all vectors in $\mathbb{R}^4$ whose dot product with each of the following vectors is 0. That is, find all $\vec{x}$ such that $\vec{x} \cdot \vec{v}_i = 0$ for $i = 1, 2, 3$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}$$

(b) Find a polynomial of degree 3 whose graph goes through the points $(2, -1)$, $(3, -59)$, $(-1, 5)$, and $(-2, -29)$.

4. Do one of the following

(a) Suppose we know that a $(2 \times 2)$ invertible matrix $A$ has all entries integers and that all the entries in $A^{-1}$ are also integers. Show that the only possible values for the determinant of $A$ are 1 and $-1$.

(b) Is it possible to have an invertible $(3 \times 3)$ matrix $A$ with $AA = O$? (Here $O$ represents the $(3 \times 3)$ zero matrix.)

5. Do one of the following

(a) Give an example of a $(2 \times 3)$ matrix $A$ and a $(3 \times 2)$ matrix $B$ for which $AB = I_2$.

(b) Suppose $A$ is a $(3 \times 3)$ matrix. Show it is always possible to find a non-zero $(3 \times 3)$ matrix $B$ with $AB = O$ where $O$ represents the $(3 \times 3)$ zero matrix. [Hint: consider the solutions of the system of equations $B\vec{x} = \vec{0}$.]