The Problems

1. Show that the function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation.

\[ T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 5x_2 \\ 0 \\ 2x_1 - 3x_2 \end{bmatrix}. \]

2. Do one of the following.

   (a) Without using technology, compute the determinant of the matrix

   \[
   \begin{bmatrix}
   0 & -1 & 0 & 1 \\
   -2 & 3 & 1 & 4 \\
   1 & -2 & 2 & 3 \\
   0 & 1 & 0 & -2
   \end{bmatrix}.
   \]

   (b) The characteristic polynomial of the matrix

   \[
   \begin{bmatrix}
   0 & 0 & 0 & 0 \\
   0 & 1 & 1 & 0 \\
   0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1
   \end{bmatrix}
   \]

   is $\lambda^2 (\lambda - 1)^2$. Find the eigenvalues and determine a basis for each eigenspace.

3. Do one of the following.

   (a) Suppose $\overrightarrow{v}$ is an eigenvector of the matrix $A$ with associated eigenvalue 3. Explain why $\overrightarrow{v}$ is also an eigenvector for the matrix $A^2 + 4I_n$. What is the associated eigenvalue?

   (b) Suppose that $A$ is a $4 \times 4$ matrix with exactly two distinct eigenvalues, 5 and $-9$ and let $E_5$ and $E_{-9}$ be the corresponding eigenspaces, respectively. Write all possible characteristic polynomials of $A$, in factored form, that are consistent with $\dim (E_5) = 1$.

4. Do one of the following.

   (a) Is the matrix $A = \begin{bmatrix} 1 & 0 \\ 10 & 2 \end{bmatrix}$ diagonalizable? If not, explain why not. If so, find an invertible matrix $S$ for which $S^{-1}AS$ is diagonal.
(b) The matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ are similar. Exhibit a matrix $S$ for which $B = S^{-1}AS$.

5. Do two of the following.

(a) Show that the set, $W = \left\{ A \in \mathbb{R}^{3 \times 3} : \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ is an eigenvector of } A \right\}$ is a subspace of $\mathbb{R}^{3 \times 3}$.

(b) Find a basis for the subspace $W = \{ A \in \mathbb{R}^{2 \times 2} : \text{trace} (A) = 0 \}$. Be sure to show that your basis both spans $W$ and is linearly independent.

(c) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $T \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$. Determine $T \begin{bmatrix} 7 \\ -11 \end{bmatrix}$.