The Problems

1. (20 points) Use mathematical induction to solve one of the following.

   (a) Let \( \phi : G \to G' \) be a group homomorphism. Prove that for any elements \( a_1, \ldots, a_k \) of \( G \),
       \[
       \phi(a_1 \cdots a_k) = \phi(a_1) \cdots \phi(a_k).
       \]

   (b) Compute the determinant of the \( n \times n \) matrix \( A_n \) given by
       \[
       A_n = \begin{bmatrix}
       1 & \cdots & 1 \\
       \vdots & \ddots & \vdots \\
       1 & \cdots & 1
       \end{bmatrix}
       \]

2. (20 points) Given a group \( G \), subgroup \( H \) of \( G \) and element \( g \in G \), we define the conjugate subgroup of \( H \) in \( G \) to be the set
   \[
   gHg^{-1} = \{ ghg^{-1} : h \in H \}.
   \]
   Prove \( gHg^{-1} \) is indeed a subgroup of \( G \).

3. (15 points) Let \( \phi : G \to G' \) be an onto homomorphism and let \( N \) be a normal subgroup of \( G \). Prove that \( \phi(N) \) is a normal subgroup of \( G' \).

4. (15 points each) Do any three of the following problems.

   (a) Prove that a group of order 30 can have at most 7 subgroups of order 5.

   (b) Let \( a, b \) be elements in a group \( G \).
       i. Suppose the product \( ab \) has finite order in \( G \). Prove the orders, in \( G \), of \( ab \) and \( ba \) are the same.
       ii. Must the orders of \( ab \) and \( ba \) be the same if the product \( ab \) has infinite order in \( G \)?

   (c) Let \( \phi : G \to G' \) be a group homomorphism with kernel \( K \). Let \( H \) be another subgroup of \( G \). Recall that \( HK = \{ hk : h \in H, k \in K \} \). Show \( \phi^{-1}(\phi(H)) = HK \).

   (d) Let \( G, G' \) be groups.
       i. What is the order of the product group \( G \times G' \)?
       ii. Let \( x \in G \) have order \( m \) and \( y \in G' \) have order \( n \). What is the order of \( (x, y) \in G \times G' \)?

   (e) Let \( S \) be a set of groups. Define the relation \( \sim \) on \( S \) by If \( G \sim H \) if and only if \( G \) is isomorphic to \( H \). Show this relation is an equivalence relation on \( S \).

   (f) Let \( \psi : G \to H \) and \( \phi : H \to K \) be homomorphisms. Thus, the composition \( \phi \circ \psi : G \to K \) is a function. Prove \( \phi \circ \psi \) is also a homomorphism. Describe the kernel of \( \phi \circ \psi \).