Mathematics 232-A  
Exam 2  
Spring 2006

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Name

Technology used:

- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.

Do any three (3) of these computational problems

C.1. Is the set of vectors $S = \{ w_1, w_2, w_3, w_4 \}$ linearly dependent or linearly independent? If it is linearly dependent, first write one of the $w$’s as a linear combination of the others and then write the set $T$ that is a subset of $S$, is linearly independent, and for which $< T >= < S >$

1. The set $S$ is linearly dependent because: the coefficient matrix for the system of equations $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 = 0$

\[
\begin{bmatrix}
7 & 2 & 4 & 1 \\
3 & 5 & 3 & 9 \\
5 & 3 & 3 & 7 \\
4 & 9 & 8 & 3 \\
2 & 7 & 6 & 1
\end{bmatrix}
\]

which has reduced row echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Since there is no leading one in the last column there are infinitely many solutions to the system and there exists a non-trivial relation of dependence for $S$.

C.2. Write each of the following complex numbers in the form $a + bi$.

(a) $i (3 - 2i) + 7 (-2 + i) = (3i + 2) + 7 (-2 - i) = -12 - 4i$

(b) $(4 - 2i) (-3 + i) = -10 + 10i$

(c) $\frac{2-i}{3+4i} = \frac{2}{25} - \frac{11}{25}i$ [Multiply by $\frac{3-4i}{3-4i}$ and simplify.]

C.3. Consider the following vectors in $C^4$.

\[
\vec{v}_1 = \begin{bmatrix}
1/2 \\
1/2 \\
1/2 \\
1/2
\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}
1/2 \\
1/2 \\
-1/2 \\
-1/2
\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}
1/2 \\
1/2 \\
1/2 \\
1/2
\end{bmatrix}
\]

Find all vectors $\vec{v}_4$ in $R^4$ so that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ form an orthonormal set.

Although you don’t need it, the formula for the Gram-Schmidt process is

\[
\vec{u}_i = \vec{v}_i - \left( \frac{< \vec{v}_i, \vec{u}_1 >}{< \vec{u}_1, \vec{u}_1 >} \right) \vec{u}_1 - \cdots - \left( \frac{< \vec{v}_i, \vec{u}_{i-1} >}{< \vec{u}_{i-1}, \vec{u}_{i-1} >} \right) \vec{u}_{i-1}
\]
2. Since the given vectors are already orthonormal we look for \( \vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \) whose inner product with each of \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) is zero. This gives us the system of equations

\[
\begin{align*}
\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d &= 0 \\
\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}c - \frac{1}{2}d &= 0 \\
\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d &= 0
\end{align*}
\]

which has solution set \( S = \left\{ d \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} : d \in \mathbb{C} \right\} \). The only vectors in this set that have norm equal to 1 are \( \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \).

C.4. The matrix \( A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \) has the property that there is at least one vector \( \vec{x} \) for which \( A\vec{x} = 5\vec{x} \). Find all such vectors.

3. \( A\vec{x} = 5\vec{x} \) can be rewritten as the system of equations

\[
\begin{align*}
4x - 2y &= 5x \\
-x + 3y &= 5y
\end{align*}
\]

and this system can be rewritten as the homogenous system

\[
\begin{align*}
-x - 2y &= 0 \\
-x - 2y &= 0
\end{align*}
\]

The solution set is: \( S = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} : x_2 \in \mathbb{C} \right\} \).

Do any two (2) of these problems from the text, homework, or class.

You may NOT just cite a theorem or result in the text. You must prove these results.

M.1. Prove that if the matrix \( A \) is nonsingular and \( B \) is any appropriately sized matrix, then \( N(AB) \subseteq N(B) \).

1. Let \( \vec{x} \) be a vector in \( N(AB) \) so that \( AB(\vec{x}) = \vec{0} \). This means \( A(B\vec{x}) = \vec{0} \) and since \( A \) is nonsingular there is only the trivial solution to this matrix equation, namely, \( B\vec{x} = \vec{0} \) which shows \( \vec{x} \in N(B) \).

M.2. Prove DMAM (Distributivity across Matrix Addition): If \( \alpha \in \mathbb{C} \), and \( A, B \in M_{mn} \), then \( \alpha (A + B) = \alpha A + \alpha B \).
2. Let $i, j$ be any indices satisfying $1 \leq i \leq m, 1 \leq j \leq n$ then

$$[\alpha (A + B)]_{ij} = \alpha [A + B]_{ij}$$

$$= \alpha (|A|_{ij} + |B|_{ij})$$

$$= \alpha |A|_{ij} + \alpha |B|_{ij}$$

$$= [\alpha A]_{ij} + \alpha |B|_{ij}$$

$$= [\alpha A + \alpha B]_{ij}$$

Since this equality holds for every entry of the two matrices, we have $\alpha (A + B) = \alpha A + \alpha B$.

M.3. Prove if $\{w_1, w_2, w_3\}$ is a linearly dependent set in $\mathbb{C}^{23}$, then the set

$$\{2w_1 + w_2 + 3w_3, -3w_1 + 2w_2 + 4w_3, w_1 + 2w_2 + 3w_3\}$$

is linearly dependent.

3. Consider the relation of linear dependence

$$\beta_1 (2w_1 + w_2 + 3w_3) + \beta_2 (-3w_1 + 2w_2 + 4w_3) + \beta_3 (w_1 + 2w_2 + 3w_3) = 0$$

which, using distributivity, commutativity and associativity can be rewritten as

$$(2\beta_1 - 3\beta_2 + \beta_3) w_1 + (\beta_1 + 2\beta_2 + 2\beta_3) w_2 + (3\beta_1 + 4\beta_2 + 3\beta_3) w_3 = 0$$

Since the set $\{w_1, w_2, w_3\}$ is linearly dependent, there is a nontrivial solution to this system of equations. That is, there are scalars $\alpha_1, \alpha_2, \alpha_3$, not all zero, that satisfy $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = 0$. Since the system of equations

$$2\beta_1 - 3\beta_2 + \beta_3 = \alpha_1$$

$$\beta_1 + 2\beta_2 + 2\beta_3 = \alpha_2$$

$$3\beta_1 + 4\beta_2 + 3\beta_3 = \alpha_3$$

has a nonsingular coefficient matrix there is a unique solution. Since at least one of the $\alpha_i$ is not zero, the $i$th equation of the system shows that at least one of the $\beta_i$’s must not be zero. Hence there is a nontrivial relation of dependence

$$\beta_1 (2w_1 + w_2 + 3w_3) + \beta_2 (-3w_1 + 2w_2 + 4w_3) + \beta_3 (w_1 + 2w_2 + 3w_3) = 0$$

and the given set $\{2w_1 + w_2 + 3w_3, -3w_1 + 2w_2 + 4w_3, w_1 + 2w_2 + 3w_3\}$ is linearly dependent.

**Do one (1) of these problems you’ve not seen before.**

T.1. Suppose $A_{n \times m}$ and $B_{m \times n}$ are matrices such that $AB = I_n$. Let $\overrightarrow{b}$ be a particular vector in $\mathbb{R}^n$.

Show that the system of equations $A\overrightarrow{x} = \overrightarrow{b}$ must be consistent.

1. Since $(AB)\overrightarrow{b} = I_n\overrightarrow{b} = \overrightarrow{b}$ then by associativity $A(B\overrightarrow{b}) = \overrightarrow{b}$ and the vector $\overrightarrow{x} = B\overrightarrow{b}$ is a solution of the matrix equation $A\overrightarrow{x} = \overrightarrow{b}$. So the corresponding system of equations must be consistent.

T.2. Use the Principle of Mathematical Induction to prove that the statement $P(n)$ given by $\sum_{k=1}^{n} (2k - 1) = n^2$ holds for all positive integers.

2. $P(1)$ is true since $\sum_{k=1}^{1} (2k - 1) = (2 - 1) = 1 = 1^2$.

Suppose $P(n)$ is true. That is, $\sum_{k=1}^{n} (2k - 1) = n^2$.

Then, $\sum_{k=1}^{n+1} (2k - 1) = [\sum_{k=1}^{n} (2k - 1)] + 2(n + 1) - 1 = n^2 + 2n + 1 = (n + 1)^2$ and the truth of $P(n + 1)$ follows from the truth of $P(n)$. Hence by the principle of mathematical induction, $\sum_{k=1}^{n} (2k - 1) = n^2$ for every positive integer $n$.  

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