1 Additional Exercises: Finite Group of Motions

1. Let $D_n$ denote the dihedral group. Express the product $x^2yx^{-1}y^{-1}x^3y^3$ in the form $x^iy^j$ in $D_n$.

2. List all the subgroups of $D_4$ and determine which are normal.

3. Find all proper normal subgroups and identify the quotient groups of the groups $D_{13}$ and $D_{15}$.

4. Prove any discrete group $G$ consisting of rotations about the origin is cyclic and is generated by $\rho_\theta$ where $\theta$ is the smallest angle of rotation in $G$.
   
   (a) Advanced Calculus students, or others interested in the completeness property of the real numbers, may wish to prove that any discrete group $G$ consisting of rotations about the origin really does have a smallest angle of rotation.

5. Let $G$ be a subgroup of $M$ that contains rotations about two different points. Prove algebraically that $G$ contains a translation.

6. Determine the point group for each of the patterns depicted in the figure on the handout labelled "Extra Exercise: Finite Group of Motions #6."

7. Prove that every discrete subgroup of $O$ is finite.

8. Prove the group of symmetries of the frieze pattern

   $\cdots \text{E E E E E E E E E E} \cdots$

   is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and an infinite cyclic group.

9. Let $G$ be the group of symmetries of the frieze pattern

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   (a) Determine the point group $\mathcal{G}$ of $G$.

   (b) For each element $\overline{g}$ of $\mathcal{G}$, and each element $g$ of $G$ which represents $\overline{g}$, describe the action of $g$ geometrically.

   (c) Let $H$ be the subgroup of translations in $G$. Determine $[G : H]$.

10. Let $G$ be a discrete group in which every element is orientation-preserving. Prove the point group $\mathcal{G}$ is a cyclic group of rotations and there is a point $p$ in the plane such that the set of group elements which fix $p$ is isomorphic to $\mathcal{G}$.

11. Let $N$ denote the group of rigid motions of the line $l = \mathbb{R}^1$. Some elements of $N$ are

    $t_a : t_a(x) = x + a$ and $s : s(x) = -x$. 

(a) Show that \{t_a, t_a s\} are all of the elements of \(N\), and describe their actions on \(l\) geometrically. [Note that \(|N|\) is infinite since there is a distinct \(t_a\) for each real number \(a\).]

(b) Compute the products \(t_a b, s t_a, ss\).

(c) Find all discrete subgroups of \(N\) which contain a translation. It will be convenient to choose your origin and unit length with reference to the particular subgroup. Prove your list is complete.

12. Prove if the point group of a lattice group \(G\) is \(C_6\), then \(L = L_G\) is an equilateral triangular lattice, and \(G\) is the group of all rotational symmetries of \(L\) about the origin.

13. Prove if the point group of a lattice group \(G\) is \(D_6\), then \(L = L_G\) is an equilateral triangular lattice, and \(G\) is the group of all symmetries of \(L\).