1 Problems

1.1 Geometry Associated with Real Positive Definite Form

1. Let \( W \) be a subspace of a Euclidean space \( V \). Prove \( W = W^\perp \).

2. Find the matrix of the projection \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) such that the image of the standard basis of \( \mathbb{R}^3 \) forms an equilateral triangle and \( T(e_1) \) points in the direction of the \( x \)-axis.

3. Let \( w \in \mathbb{R}^n \) be a vector of unit length.

   (a) Prove the matrix \( P = I - 2ww^t \) is orthogonal.

   (b) Prove multiplication by \( P \) is a reflection through the space \( W^\perp \) orthogonal to \( w \). That is, prove if we write an arbitrary vector \( v = cw + w' \) where \( w' \in W^\perp \), then \( Pv = -cw + w' \).

   (c) Let \( X,Y \) be arbitrary vectors in \( \mathbb{R}^n \) with the same length. Determine a vector \( w \) such that \( PX = Y \). [Hint: draw generic \( X + Y \) and \( X - Y \).]

4. Use the above problem (number 3) to prove every orthogonal \( n \times n \) matrix is a product of at most \( n \) reflections.

1.2 Hermitian Forms

1. Prove a matrix \( A \) is hermitian if and only if the associated form \( X^*AX \) is a hermitian form.

2. Is \( \langle X,Y \rangle = x_1y_1 + ix_1y_2 - ix_2y_1 + ix_2y_2 \) on \( \mathbb{C}^2 \) a hermitian form?

3. Prove the determinant of a hermitian matrix is a real number.

4. Let \( P_n \) be the vector space of polynomials of degree less than or equal to \( n \).

   (a) Show

   \[
   \langle f, g \rangle = \int_0^{2\pi} f(e^{i\theta})g(e^{i\theta}) \, d\theta
   \]

   is a positive definite hermitian form on \( P_n \).

   (b) Find an orthonormal basis for this form when \( n = 3 \).

5. Determine whether or not the following rules define hermitian forms on the space \( \mathbb{C}^{m \times n} \) of complex matrices and, if so, determine their signature.

   (a) \( \langle A, B \rangle = \text{Trace}(A^*B) \).
(b) $\langle A, B \rangle = \text{Trace} (\bar{AB})$.

1.3 Spectral Theorem

1. 

(a) Find a unitary matrix $P$ so that $PAP^*$ is diagonal when 

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

(b) Find a real orthogonal matrix $P$ so that $PAP^t$ is diagonal when 

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2. Prove a real symmetric matrix is positive definite if and only if all of its eigenvalues are positive.

1.4 Conics and Quadrics

1. Determine the type of the quadric 

$$x^2 + 4xy + 2xz + z^2 + 3x + z - 6 = 0.$$ 

2. 

(a) Describe the types of conic in terms of the signature of the quadratic form. 
(b) Do the same for quadrics in $R^3$. 