1 Problems

1. In class, we outlined a process (using the substitution principal) that shows, for any ring \( R, R[xy] \approx R[x][y] \). Fill in the details of that process or fill in the details of the following alternative.

   (a) Extend \( R \rightarrow R[x][y] \) to a map \( \Phi : R[x,y] \rightarrow R[x][y] \)
   (b) Extend \( R[x] \rightarrow R[x,y] \) to a map \( \Psi : R[x][y] \rightarrow R[x,y] \)
   (c) Use uniqueness of extension to show \( \Phi \Psi \) and \( \Psi \Phi \) are both the identity maps. (This shows \( \Phi \) is an isomorphism).

2. Do all of the following

   (a) For which integers \( n \) does \( x^2 + x + 1 \) divide \( x^4 + 3x^3 + x^2 + 6x + 10 \) in \( (\mathbb{Z}/n\mathbb{Z})[x] \)?
   (b) Describe the kernel of the map defined by \( \phi : \mathbb{Z}[x] \rightarrow \mathbb{R} \) given by \( \phi(f(x)) = f(1 + \sqrt{2}) \).

3. Prove

   (a) the kernel of the homomorphism \( \phi : \mathbb{C}[x,y] \rightarrow \mathbb{C}[t] \) given by \( \phi(f(x,y)) = f(t^2, t^3) \) is the principal ideal generated by the polynomial \( y^2 - x^3 \).
   (b) describe the image of \( \phi \) explicitly.

4. Let \( I, J \) be ideals of a ring \( R \).

   (a) Show by example that \( I \cup J \) need not be an ideal but show the set \( I + J = \{ r \in R : r = x + y, x \in I, y \in J \} \) is an ideal. This ideal is called the sum of \( I \) and \( J \).
   (b) Prove that \( I \cap J \) is an ideal.
   (c) Show by example that the set of products \( \{ xy : x \in I, y \in J \} \) need not be an ideal but that the set of finite sums \( \sum_{i,j} x_i y_j \) of products of elements of \( I \) and \( J \) is an ideal. This ideal is called the product ideal and is denoted \( IJ \).
   (d) Prove \( IJ \subset I \cap J \).
   (e) Show by example that \( IJ \) and \( I \cap J \) need not be equal.